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# The New Concept of Reversibility for Viscoelastic Maxwell Class Models

Abstract

Assuming Boltzmann's superposition rule and non-aging materials the reversibility of linear phenomenological viscoelastic models are investigated with special attention to Maxwell-Burgers bodies like.

On the basis of Terzaghi's porous ground and hysteresis phenomenon, the new concept of reversibility is presented. The above idea is illustrated in the cases of Maxwell and Kelvin-Voigt models.

Finally, the task of simple beam vibration is analyzed, while its material properties are found as adequate to Maxwell's model. The result that has been obtained is proving that Maxwell model, and generalizing Maxwell Class models, are reversible.

## 1. Introduction

Not frequently, but it happens that in technical literature, especially in highway engineering, there appears incorrect classification of Burgers model [1], [2], [3]. As it is presented at Fig. 1., Burgers creep function is an adequate approximation curve for describing presented laboratory results, but the straight application yield rather to large least squares method error [4]. The components of Burgers model can be sectioned off into following elements –

$$\uparrow \varepsilon_B = \varepsilon_e + \varepsilon_{v-e}^{\text{linear}} + \varepsilon_{v-e}^{\text{exponental}}, \quad \downarrow \varepsilon_B = \varepsilon_e + 0 + \varepsilon_{v-e}^{\text{exponential}}, \tag{1}$$

where $\varepsilon_e$ - stands for the immediate elastic strain, $\varepsilon_{v-e}^{\text{linar}}$ - linear viscous flow process of strain, $\varepsilon_{v-e}^{\text{exponential}}$ - exponential viscous flow; the arrows orientation is associated with - $\uparrow$ - applied load period and $\downarrow$ - destressing.

The above description strongly shows the absence of $\varepsilon_{v-e}^{\text{linar}}$ component in destresing process. The lack of this element entails treating the residual value in Burgers creep function as an irreversible part and as a consequence to classify Burgers model as irreversible too.

Our aim is to show the contradiction of such concept and try to formulate the idea of reversibility.

With reference to straightforward applying of Burgers model – is better to use the expanded Burgers model by series connection with Saint-Venant element [5] –

$$\varepsilon_\Sigma = \varepsilon_B + \varepsilon_{St-V} \tag{2}$$

where $\varepsilon_\Sigma$ - summary strain, $\varepsilon_B$ - Burgers model strain, $\varepsilon_{St-V}$ - plastic strain., in that case we bear in mind that the involved model belongs to Bingham Class bodies. In such case, one can avoid the inaccuracy of approximation errors.

In further considerations Maxwell Class model is understood as - the Maxwell as well as Gubanow and Burgers and similar other models.

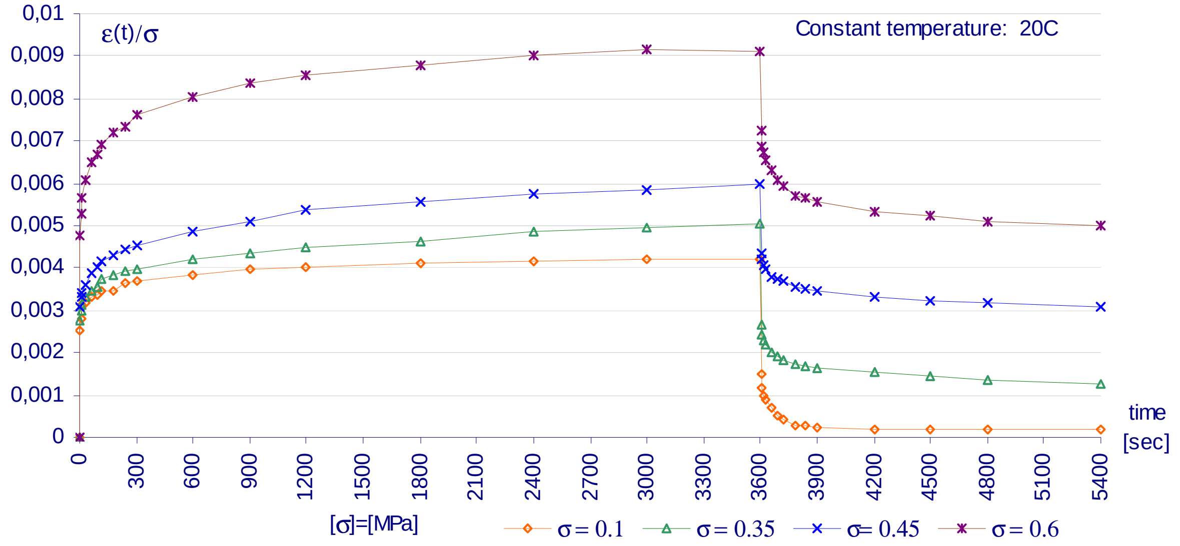


Fig. 1. An example of typical road pavement asphalt-mortar creep test graph. The ordinates related to the last abscissa value, in technical sense, are treated as measures of plastic deformations.

## 2. Viscoelasticity

Viscoelesticity could be treated as a natural extension of elasticity. The elasticity is precisely defined by mathematical relations and physical concepts. In the case of viscoelasticity the definition is not so clear. We can define viscoelasticity as a relation between stress and strain tensors and their time derivatives of different ranks

$$\Phi\left(I_m^{(\sigma)}, I_n^{(\varepsilon)}, \dot{I}_m^{(\sigma)}, \dot{I}_n^{(\varepsilon)}, , \ddot{I}_m^{(\sigma)}, \ddot{I}_n^{(\varepsilon)}, ...., t\right) \rightarrow F(\sigma, \varepsilon, \dot{\sigma}, \dot{\varepsilon}, \ddot{\sigma}, \ddot{\varepsilon}, , ...., t) = 0, \quad (3)$$

where $I_m^{(\sigma)}$, $m = 1,2,3$ - invariants of the stress tensor $\sigma$, $I_n^{(\varepsilon)}$, n=1,2,3 - invariants of the strain tensor, dots which can be noticed over the symbols means partial derivatives according to time.

The second element of viscoelastisity definition comes from Boltzmann's superposition, which refers to the load history. We conclude that:

- Each load is a component of the load history,
- The total final strain is a sum of deformations caused by all components of the load history.

Although the examples of the Bolzmann's principle can be found in many works, we illustrate them below by taking into account the load (4) acting to Kelvin-Voigt and Burgers models –

$$\sigma = \sigma_1[H(t - t_1) - H(t - t_4)] + \sigma_2[H(t - t_2) - H(t - t_3)] \quad (4)$$

where: (5) $H(t - t_{(.)}) = \begin{cases} 1 \leftarrow (t - t_{(.)}) \geq 0 \\ 0 \leftarrow (t - t_{(.)}) < 0 \end{cases}$ - Heavisidea's step function,

$\sigma_1$, $\sigma_2$; $sig[\sigma_1] = sig[\sigma_2]$ - different stress values, $t_1 < t_2 < t_3 < t_4$ - time moments.

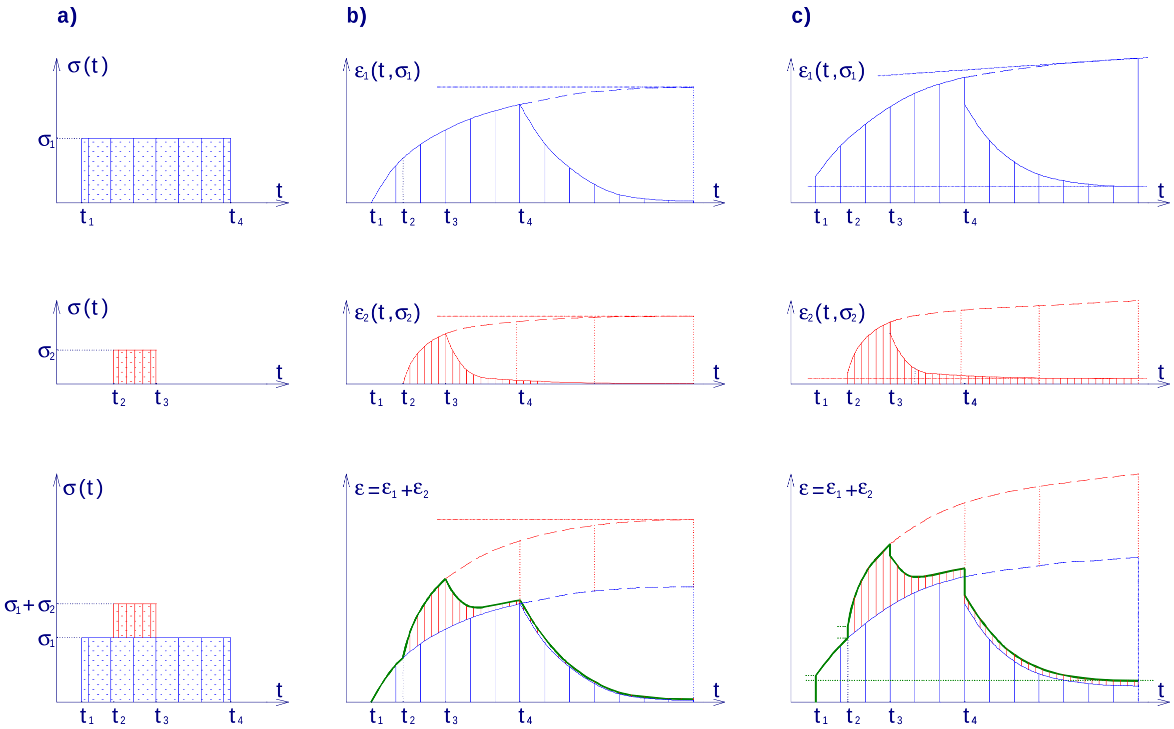


Fig. 1. Boltzmann's superposition. a) Loads. b) Kelvina-Voigt model reaction. c) Burgers body response.

In the case b) the unloading causes that the deformation fades after longenough time, so we get

$$\lim_{t\to\infty}\varepsilon_1 = \lim_{t\to\infty}\varepsilon_2 \to 0 \ , \qquad (6)$$

whereas for Burgers material c), according to the same load history a), we observe permanent strain value when $t \to \infty$:

$$\lim_{t\to\infty}\varepsilon_1 = \varepsilon_{1\infty} \neq 0\,, \qquad \lim_{t\to\infty}\varepsilon_2 = \varepsilon_{2\infty} \neq 0\,, \qquad \lim_{t\to\infty}\varepsilon = \varepsilon_{1\infty} + \varepsilon_{2\infty} = \varepsilon_{\infty}\,. \qquad (7)$$

According to Fig. 1., for $t \to \infty$: Kelvina-Voigt model return to the initial null strain value while Burgers model achieve a residual (non-initial) strain value and looks like to be irreversible one, as has been noticed in the beginning of the paper. All the time we have in mind that both of these models are linearly viscoelastic.

Let us also discuss the concept of Perzyna [6]. In spite of the fact that he regards visco-plasticity, his methodology is very useful for adopting to our problem. We can read "*the methods of viscoplasticity belong neigher to rheology nor to plasticity*". On that basis he introduces two material classes:

- *Elastic - viscoplastic* – for which the material shows viscous properties in both elastic and plastic regions.

- *Elastic / viscoplastic* - reserved for materials showing viscous properties in plastic region only.

In both cases above the viscous flow deformation could be further divided into irreversible and reversible parts

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{v} + \varepsilon_{ij}^{p} \ , \qquad (8)$$

since $\varepsilon_{ij}$, $\varepsilon_{ij}^{e}$, $\varepsilon_{ij}^{v}$, $\varepsilon_{ij}^{p}$ - are as it follows: total, elastic, viscous and plastic strain tensor components. Contemporary both processes are classified as Bingham model.

Adopting Perzyna's concept we can introduce two types of viscoelasticity:

- *Visco - elasticity* – when viscous and elastic components develop in parallel and
- *Visco / elasticity* – for the previous immediate elastic deformation and after that viscous deformation process,

but both of them are reversible. According to the time scale of road pavement asphalt-mortar creep test, it could be included into *visco / elasticity* class, which is identical with viscoelasticity in further considerations.

Relation (1) can be treated as a complete constitutive relation and, in our consideration, we confine ourselves to invariant linear processes i.e. satisfying the relation –

$$\varepsilon(\sigma, t) = \sum_{m=1,2,3}^{k_m} \sigma_m \, f\left[1, \left(t - t_m\right)\right], \qquad (9)$$

where $\varepsilon$ -strain, $\sigma_{(,)}$ -stress in the one dimensional case and $t$ – is a time parameter.

We would like to emphasize that the creep problem is reduced to the case of one dimensional body and to the analysis of an initial stage – transient creep. We disregard steady state creep and creep to failure. Additionally, we assume that initial conditions are always homogenous.

3. Maxwell Class Models

Let us compare the differential constitutive relations of several elementary viscoelastic models:

- Maxwell $\dot{\sigma} a_1 + \sigma a_0 = \_ + \dot{\varepsilon} b_1$, (10.1)
- Burgers $\ddot{\sigma} a_2 + \dot{\sigma} a_1 + \sigma a_0 = \_ + \dot{\varepsilon} b_1 + \ddot{\varepsilon} b_2$, (10.2)
- Kelvina-Voigt $\sigma a_0 = \varepsilon b_0 + \dot{\varepsilon} b_1$, (11.1)
- Zener $\dot{\sigma} a_1 + \sigma a_0 = \varepsilon b_0 + \dot{\varepsilon} b_1$; (11.2)

where $a_0, a_1, a_2, b_0, b_1, b_2$ - are the material characteristics. In spite of the same notation they can have different values for different models.

Nowacki [7] comments the following formula $P(D)\sigma(t) = Q(D)\varepsilon(t)$ by sentences – '*The ranks of Q(D) and P(D) operators should be equal (m=n)*' and '*It is admissible to assume the case when the rank of Q(D)is one rank higher then the P(D) rank*'.

Also Rzanicyn analyses the problem [8] writing following remarks – *'The rank of differential equation is equal to the number of dashpots in the model'* and *'In case of lacks of constant elastic connection between deforming points, then instead of* $\varepsilon$ *we have its time derivative. The long-standing stress yields the* $\varepsilon$ *increase up to infinity'*.

For (10.1-2) and (11.1-2) the conditions formulated by Nowacki and Rzanicyn are fulfilled but we can also notice not discussed feature i.e. the absence of one sequence element in Maxwell and Budgers models, notified by '_' in (11.1-2) formulae.

Now we can also define the Maxwell Class as models for which theirs constructive equations are of type (26) and in general as the materials described in following way –

$$\begin{aligned} &\overset{(n)}{\sigma} a_n + \_ + \overset{(n-2)}{\sigma} a_{n-2} + \ldots + \ddot{\sigma} a_2 + \_ + \sigma a_0 = \\ &\qquad = \_ + \dot{\varepsilon} b_1 + \ddot{\varepsilon} b_2 + \ldots + \overset{(m-3)}{\varepsilon} b_{m-3} + \_ + \overset{(m-1)}{\varepsilon} b_{m-1} + \overset{(m)}{\varepsilon} b_m \end{aligned} \tag{12}$$

In mathematical sense, the absence of $\varepsilon b_0$ element in relation (11.1) or (11.2) is the reason of so-called irreversibility of those models.

Having in mind that initial conditions are homogenous, we can solve the differential equation obtaining zero values for all constants. Let us assume that the load function for a Maxwell model is of the form –

$$\sigma = \sigma_o [H(t - t_1) - H(t - t_2)], \quad t_1 < t_2 , \tag{13}$$

then we arrive at following solution –

$$\varepsilon(t) = \eta(t - t_1)\left[\frac{1}{E} + \frac{1}{\eta}(t - t_1)\right] - \eta(t - t_2)\left[\frac{1}{E} + \frac{1}{\eta}(t - t_2)\right] \qquad \text{(see (20))}$$

where for $t > t_2$ we obtain

$$\varepsilon(t) = \frac{1}{\eta}\left(t_2 - t_1\right) \text{ - permanent value.} \tag{14}$$

We use 'permanent value' to distinct above result, which is the solution to differential equation –

$$0 = \dot{\varepsilon} \rightarrow \varepsilon\left(t\right) = \chi \ \text{ since } \chi \in R \ \text{ and } t > t_2 , \tag{15}$$

where the value of differential equation constant $C = 0$ coming from initial conditions.

Concluding – *permanent value is caused by the load function and the model has not to be irreversible.*

Ending this part we repeat some information of Maxwell model. We use the classical notation. Viscoelastic behavior can be simulated by spring – elastic resistance $E$ and viscous dashpot characterized by $\eta$ and which are connected together in series way.

The customary differential definition form is as follows –

$$\frac{d\varepsilon}{dt}=\frac{1}{E}\frac{d\sigma}{dt}+\frac{1}{\eta}\sigma \text{ or } \dot{\varepsilon}=\frac{\dot{\sigma}}{E}+\frac{\sigma}{\eta}. \tag{16}$$

Applying the Carsona-Heaviside'a transformation [9] we obtain -

$$\tilde{\varepsilon}=\frac{(p\tilde{\sigma})}{p}\left(\frac{1}{E}+\frac{1}{p\eta}\right)\to\varepsilon=\int_{\theta=0}^{\theta=t}\dot{\sigma}(\theta)\left(\frac{1}{E}+\frac{1}{\eta}(t-\theta)\right)d\theta\ . \tag{17}$$

Involving Dirac's impulse –

$$\int_{-\infty}^{\infty}\delta(t-t_o)f(t)\,dt\overset{def.}{=}H(t-t_o)f(t_o),\ (13)\qquad \frac{d}{dt}[H(t-t_o)]\overset{def.}{=}\delta(t-t_o)\,. \tag{18}$$

Subsequently, we get –

$$\dot{\sigma}(t)=\sigma_o[\delta(t-t_1)-\delta(t-t_2)], \tag{19}$$

$$\varepsilon(t)=\eta(t-t_1)\left[\frac{1}{E}+\frac{1}{\eta}(t-t_1)\right]-\eta(t-t_2)\left[\frac{1}{E}+\frac{1}{\eta}(t-t_2)\right]. \tag{20}$$

Dividing time abscissa into sectors, extreme points occur –

$$\lim_{t\to{}_{-}t_2}\varepsilon(t)=\frac{1}{E}+\frac{1}{\eta}({}_{-}t_2-t_1)\to\frac{1}{E}+\frac{1}{\eta}(t_2-t_1)\ , \tag{21.1}$$

$$\lim_{t\to{}_{+}t_2}\varepsilon(t)=\left[\frac{1}{E}+\frac{1}{\eta}({}_{+}t_2-t_1)\right]-\left[\frac{1}{E}+\frac{1}{\eta}({}_{+}t_2-t_2)\right]\to\frac{1}{\eta}(t_2-t_1)=\lim_{t\to{}_{>}t_2}\varepsilon(t)\ , \tag{21.2}$$

if ${}_{-}t_2$, ${}_{+}t_2$ differs infinitely from $t_2$ appropriately approaching from left and right side to $t_2$ and ${}_{>}t_2 > t_2$.

Fig. 3. is an illustration of creep function according to Maxwell model.

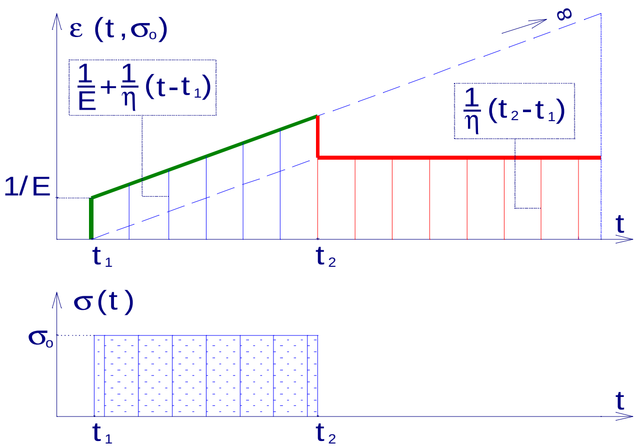


Fig. 3. The deformation process in the cases of Maxwell model and load function (13)

4. The new concept of reversibility for Maxwell Class materials

In the beginning, let us recall two simple examples of physical processes.

First is thanks to so-called '*Terzaghi's porous ground*' [10] also known as '*Terzaghi's foam*'. He used successfully the following sequence to estimate settlement of the buildings:

- The upper surface of ground in natural condition is loaded by erected building; at first we can observe immediate deflection caused be elastic ground structure response, next
- The retarding and decaying flow is observed due to water dislodging, which was filling ground porous.

Let us elongate this concept, adding two new stages –

- The building structure is disassembled which involves immediate response of ground elastic structure
- After longenough time we can also notice the recovering flow which could be caused by gravity forces. In such process, water can fill destressed ground porous volumes again.

If human live is properly long, one can probably prove above story.

The second example is extremely trivial. We consider three configurations of typical car shock absorber the structure of which is practically identical with Newtonian element in Maxwell model (Fig. 4.). Assuming an arbitrary configuration as initial - $K_p$ we push the device with $P_1$ force. That deforms the absorber to actual configuration $K_{a1}$ (Fig. 4.a.). Sequentially, applying force $-P_2$ we pull the absorber ($P_2$ could be equal or not to $P_1$) achieving the $K_{a2}$ configuration (Fig. 4.b.).

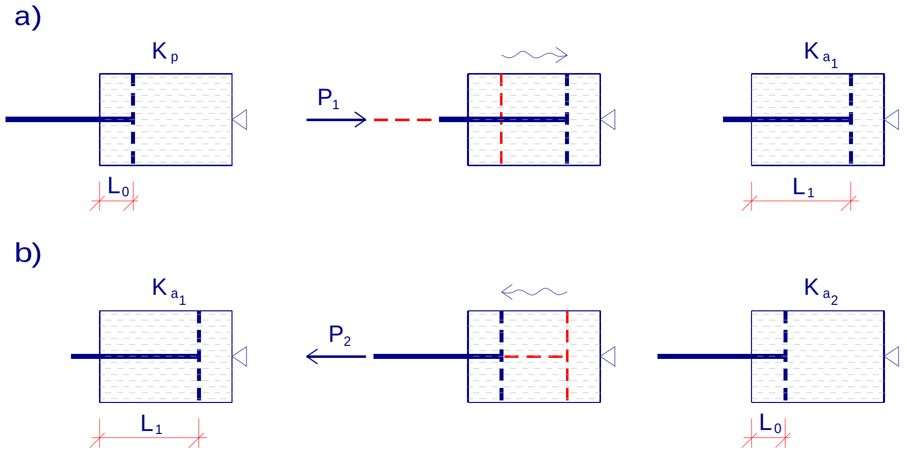


Fig. 4. The sequence of Newton body model configurations

Now we have to answer the aroused question: are the configurations $K_{a1}$ and $K_{a2}$ equal or not ?

In geometrical meaning they are equal without doubt.

In physical sense the sets – {dash pot structure, work, energy} – of $K_{a1}$ and $K_{a2}$ can be treated in many situations as convergent or even equal.

Fulfilling the superposition rule, reversibility of Maxwell Class models can be better understood with the help of hysteresis. We can fully adopt all features which come as a result of this notion i.e.: *hardening*, *softening*. Yet, the most creative element is the advent of the alternate sequence of loads and anti-loads which generate the loop from initial and back to initial configuration.

We should also remember that our nearest concept – Bauschinger hysteresis is almost always presented as a relation of $\varepsilon \sim \sigma$ coordinates.

On the basis of above analysis and involved assumptions, we can propose the following new definition of viscoelastic reversibility –

- *Since the sequence of loads and anti-loads is defined (it could be called fundamental) and*
- *If after applying the sequence, the residual deformation is observed – the material belongs to Bingham Class,*
- *If contrary – the material is viscoelastc and is characterized as entirely reversible.*

Let us now build the fundamental sequence for Maxwell and Kelvin-Voigt models assuming –

$$\sigma = \sigma_o\{[H(t-t_1)-H(t-t_2)]-[H(t-t_3)-H(t-t_4)]\}\,, \qquad (22)$$

since $t_1 < t_2 < t_3 < t_4$. Fig. 5. shows the graph of models reversibility in details according to (22).

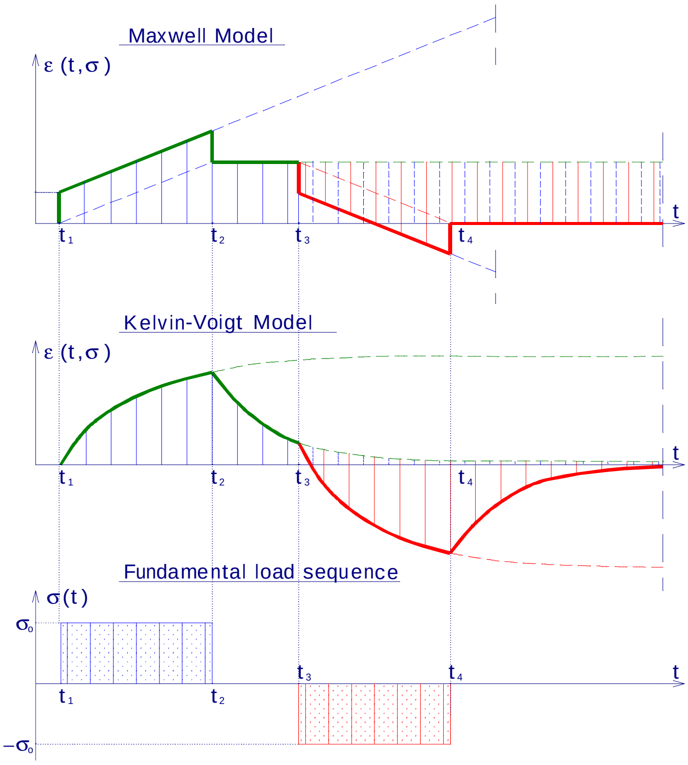


Fig. 5. Reversibility of viscoelastic models, including Maxwell Class too.

## 5. The Vibration of Simply Supported Maxwell Beam

That subject was investigated by many authors, for example Nowacki [5] looked for common effects comparing elasticity and viscoelasticity in Maxwell and Kelvin-Voigt models. The topic of our consideration is to examine the vibration process at the infinite time moment.

The dynamical equilibrium equation for elastic infinitesimal beam element is as follows –

$$\left[\mathrm{EJ}\frac{\partial^4}{\partial x^4}+\frac{\gamma A}{g}\frac{\partial^2}{\partial t^2}\right]w(x,t)=q_1(x,t)\ , \tag{23}$$

where: w-beam deflection, A-constant beam cross-section, EJ-bending rigidity, $q_1$-load linear density.
Denoting beam span by L, we introduce dimensionless coordinates –

$$x=\xi L\ ,\ w=\omega L\ ,\ t=t_o\tau\ ; \tag{24}$$

where $t_o$ - positive constant with time units,
that involves –

$$\frac{\mathrm{EJ}}{(L)^3}\omega^{IV}(\xi,\tau)+\frac{\gamma AL}{g(t_o)^2}\ddot{\omega}(\xi,\tau)=q_1(\xi,\tau)\ , \tag{25}$$

since: $\gamma$ - material weight density, g – gravity acceleration, $\omega^{IV}$ - fourth order derivative according to $\xi$, $\ddot{\omega}$ - the second rank dimensionless time parameter derivative.
Assuming:

$$(t_o)^2=\frac{\gamma A(L)^4}{Jg}\ \text{ and }\ q=q_1\frac{(L)^3}{J} \tag{26.1-2}$$

we obtain –

$$E\,\omega^{IV}+\ddot{\omega}=q\ . \tag{27}$$

With the help of Carson-Heaviside transformation according to $\tau$, and involving Alfrey's analogy we can turn into viscoelastic problem –

$$\tilde{E}\,\tilde{\omega}^{IV}+(p)^2\tilde{\omega}=\tilde{q}\ , \tag{28}$$

since the initial conditions are homogenous. In the case of Maxwell model we obtain –

$$\tilde{E}=p\frac{1}{E}+\frac{1}{\eta}\ , \tag{29}$$

where $\tilde{E}$ is not Young modulus (E) transformation.
Let the load function be Dirac's impulse –

$$q=q_\xi q_\tau=\delta\left(\xi-\frac{1}{2}\right)\delta(\tau-\tau_o) \tag{30}$$

that yield

$$\tilde{q} = q_\xi \tilde{q}_\tau \quad . \tag{31}$$

We look for the solution expanding the unknowns and load function into Fourier sine series according to $\xi$ -

$$\tilde{\omega}(\xi, p) = \sum_{j=1,2,\ldots}^{\infty} \tilde{\omega}_j(p) \sin j\pi\xi \,, \quad \tilde{q}(\xi, p) = \tilde{q}_\tau(p) \sum_{j=1,2,\ldots}^{\infty} \left(q_\xi\right)_j \sin j\pi\xi \ , \tag{32.1-2}$$

$$\left(q_\xi\right)_j = 2\int_0^1 \delta\left(\xi - \frac{1}{2}\right) \sin j\pi\xi \, d\xi = 2\sin\frac{j\pi}{2} \ . \tag{33}$$

By virtue of series properties, we have –

$$\underset{j=1,2,\ldots,\infty}{\forall} \ \tilde{\omega}_j = \frac{\tilde{q}_\tau \left(q_\xi\right)_j}{(p)^2 + \tilde{E}(j\pi)^4} = \frac{\tilde{q}_\tau \left(q_\xi\right)_j}{(p)^2 + p\frac{(j\pi)^4}{E} + \frac{(j\pi)^4}{\eta}} = \frac{\tilde{q}_\tau \left(q_\xi\right)_j}{\tilde{f}(p) = (p - p_1)(p - p_2)} . \tag{34}$$

The roots of $\tilde{f}(p) = 0$ are as follows –

$$p_{1,2} = -\frac{(j\pi)^4}{2E}\left(1 \mp \sqrt{1 - \frac{1}{\eta}\left[\frac{2E}{(j\pi)^2}\right]^2}\right) = \alpha\left(1 \mp \sqrt{\Delta}\right) . \tag{35}$$

Applying faltung theorem we can find Fourier coefficients $\omega_j$ –

$$\omega_j = \left(q_\xi\right)_j \int_0^\tau \delta(\theta - \tau_o)\, C^{-1}\left[\frac{p}{(p - p_1)(p - p_2)}\right](\tau - \theta)\, d\theta = \tag{36}$$

$$= \left(q_\xi\right)_j H(\tau - \tau_o)\, C^{-1}\left[\frac{p}{(p - p_1)(p - p_2)}\right](\tau - \tau_o) ,$$

where $C^{-1}[\cdot]$ is the symbol of Carson-Heavisidea retransformation.

Both roots (35) are $j$ function. Having in mind that $E$ and $\eta$ are positive we arrive at –

$$\lim_{j\to\infty} \sqrt{\Delta} = \lim_{j\to\infty} \sqrt{1 - \frac{1}{\eta}\left[\frac{2E}{(j\pi)^2}\right]^2} = 1 \ , \tag{37}$$

and in consequence for large enough $j$ value the roots are real and equal to –

$$p_{1\infty} = 0 \,, \qquad p_{2\infty} = -\frac{(j\pi)^4}{E} \ . \tag{38.1-2}$$

We have to analyze three potential variants –

I. $\Delta > 0 \rightarrow 1 > \frac{1}{\eta}\left[\frac{2E}{(j\pi)^2}\right]^2 \rightarrow$ the roots are real and negative, $p_1 < p_2$ ,

II. $\Delta < 0 \to 1 < \frac{1}{\eta}\left[\frac{2E}{(j\pi)^2}\right]^2$ $\to$ the roots are conjugative complex $p_1 = \overline{p}_2$ when $\mathrm{Re}(p_1) < 0$ ,

III. $\Delta = 0 \to 1 = \frac{1}{\eta}\left[\frac{2E}{(j\pi)^2}\right]^2$ $\to$ we have dual real root $p_1 = p_2 = -\frac{(j\pi)^4}{2E}$, it could be only for one $j$ index value.

The complexity of the problem consists in simultaneous occurrence of all (I-III) variants. Simplifying, let us assume that we found $j_{gr.}$ by solving III and $j_{gr.} \notin N$. $j_{gr.}$ is dividing $j$ domain into two parts where -

- Variant I is valid for $_{-}j < j_{gr.}$ and
- Variant II when $_{+}j > j_{gr.}$.

Additionally, we can state $j_{gr.}$ is not large and we can neglect the condition (38.1-2) which obeys $j \to \infty$.

Variant I is associated with hard viscous damping. Variant II describes decaying beam vibration.

In our problem the Jordan's lemma is fulfilled and we can apply residual theorem. The original for the variant I has the form -

$$C^{-1}\left[\frac{p}{(p-p_1)(p-p_2)}\right] = \sum \mathrm{Res}\frac{e^{p\tau}}{\left[p-\alpha\left(1-\sqrt{\Delta}\right)\right]\left[p-\alpha\left(1+\sqrt{\Delta}\right)\right]} = \frac{e^{\alpha\tau}}{\alpha\sqrt{\Delta}}\mathrm{sh}\left(\alpha\sqrt{\Delta}\,\tau\right). \quad (39)$$

For variant II, with the help of Euler's formulae, we arrive at –

$$C^{-1}\left[\frac{p}{(p-p_1)(p-p_2)}\right] = C^{-1}\left[\frac{p}{\left[p-\alpha\left(1-i\sqrt{|\Delta|}\right)\right]\left[p-\alpha\left(1+i\sqrt{|\Delta|}\right)\right]}\right] =$$

$$= \sum \mathrm{Res}\frac{e^{p\tau}}{\left[p-\alpha\left(1-i\sqrt{|\Delta|}\right)\right]\left[p-\alpha\left(1+i\sqrt{|\Delta|}\right)\right]} = e^{\alpha\tau}\frac{\sin\left(\alpha\sqrt{|\Delta|}\tau\right)}{\alpha\sqrt{|\Delta|}}, \quad (40)$$

where $i = \sqrt{-1}$ .

In case of large $j$ values we have –

$$\lim_{j\to\infty}\Delta \to 1 \text{ and } \lim_{j\to\infty}\left(e^{\alpha\tau}\frac{\sin\left(\alpha\sqrt{\Delta}\tau\right)}{\alpha\sqrt{\Delta}}\right) \to 0. \quad (41)$$

On the basis of (46) we get –

$$\omega_{_{-}j} = 2\sin\frac{_{-}j\pi}{2}H(\tau-\tau_o)\frac{e^{\alpha(\tau-\tau_o)}}{\alpha\sqrt{\Delta}}\mathrm{sh}\left[\alpha\sqrt{\Delta}(\tau-\tau_o)\right] \quad (42)$$

and

$$\omega_{{}_{+}j} = 2\sin\frac{{}_{+}j\pi}{2} H(\tau-\tau_o)\frac{e^{\alpha(\tau-\tau_o)}}{\alpha\sqrt{|\Delta|}}\sin\left[\alpha\sqrt{|\Delta|}(\tau-\tau_o)\right]. \tag{43}$$

The solution of the problem has the following series form –

$$\omega(\xi,\tau) = \sum_{{}_{-}j=1,2,\ldots}^{\mathrm{Int}(j_{gr.})} \omega_{{}_{-}j}\sin({}_{-}j\pi\xi) + \sum_{{}_{+}j=\mathrm{Int}(j_{gr.})+1,\ldots}^{\infty} \omega_{{}_{+}j}\sin({}_{+}j\pi\xi). \tag{44}$$

Now we can find the limit of (54) for $\xi = \frac{1}{2}$ and $\tau \to \infty$ –

$$\lim_{\tau\to\infty}\omega\left(\frac{1}{2},\tau\right) = 2\lim_{\tau\to\infty}\left\{\sum_{{}_{-}j=1,2,\ldots}^{\mathrm{Int}(j_{gr.})}\left(\sin\frac{{}_{-}j\pi}{2}\right)^2\frac{e^{\alpha(\tau-\tau_o)}}{\alpha\sqrt{\Delta}}\mathrm{sh}\left[\alpha\sqrt{\Delta}(\tau-\tau_o)\right]+\right.$$
$$\left.+\sum_{{}_{+}j=\mathrm{Int}(j_{gr.})+1,\ldots}^{\infty}\left(\sin\frac{{}_{+}j\pi}{2}\right)^2\frac{e^{\alpha(\tau-\tau_o)}}{\alpha\sqrt{|\Delta|}}\sin\left[\alpha\sqrt{|\Delta|}(\tau-\tau_o)\right]\right\} = 0. \tag{45}$$

The result that has been obtained is clearly proving that Maxwell model, and generalizing Maxwell Cass models, are reversible.

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